cts functions on intervals

Let $A \subseteq \mathbb{R}$ be an interval (non-degenerate) and
Let $f: A \to \mathbb{R}$ be cts.

Thm. Suppose $f: A = [a,b] \to \mathbb{R}$ cts. Then
(i) f is bounded (above and below) [Boundedness Th]

(ii) $\exists\ x_*, x^* \in [a,b]$ s.t. $f(x_*) \leq f(x) \leq f(x^*)\ \forall x$.
(max-min value Th)

Pf (i) We shall show only that f is bounded above.
If not then, $\forall\ n \in \mathbb{N}\ \exists\ x_n \in [a,b]$ s.t
$f(x_n) > n$. Then (??), $\exists$ a subsequence
$x_{n_k} \to \bar{x}$ for some $\bar{x} \in [a,b]$. Hence (?)
$f(x_{n_k}) \to f(\bar{x}) \in \mathbb{R}$ and so $(f(x_{n_k}))$ is
bounded, contradicting the ineq

$$f(x_{n_k}) > n_k \geq k \quad \forall\ k \in \mathbb{N}.$$

(ii) Let $\sup\{f(x) : x \in [a,b]\}$ be denoted by
S ($\in \mathbb{R}$ by (i)). Then, $\forall\ n \in \mathbb{N}$, $\exists\ x_n \in [a,b]$ s.t

$$S - \frac{1}{n} < f(x_n) \qquad (*)$$

Similar as in (i), $\exists$ subseq $(x_{n_k}) \to x^* \in [a,b]$
Then $f(x_{n_k}) \to f(x^*)$ and it follows (??) from $(*)$
that $S = \lim_k \left(S - \frac{1}{n_k}\right) \overset{?}{\underset{\text{why}}{\leq}} \lim_k f(x_{n_k}) = f(x^*)$
and, since $f(x) \leq S\ \forall x$, then we have $f(x) \leq f(x^*)$.

Th2 Let A be an interval and $f : A \to \mathbb{R}$ cts.

(i) Suppose $\exists$ $a, b \in A$ with $f(a) f(b) < 0$. Then $\exists$ c lying between a and b such that $f(c) = 0$

(ii) Suppose $\exists$ $a, b \in A$ and $\alpha \in \mathbb{R}$ s.t. $f(a) < \alpha < f(b)$. Then $\exists$ c - - - - - - - - - - - - - - $f(c) = \alpha$.

Proof. (ii) follows from (i) easily (Ex).

To prove (i), we assume wlg that $a < b$ and $f(a) < 0 < f(b)$. Assume $\nexists$ $c \in (a, b)$ s.t. $f(c) = 0$. We will have a contradiction. Indeed, we compute $f\left(\frac{a+b}{2}\right)$; if $f\left(\frac{a+b}{2}\right) < 0$ we let

$$[a_1, b_1] = \left[\frac{a+b}{2}, b\right],$$

if $f\left(\frac{a+b}{2}\right) > 0$ we let

$$[a_1, b_1] = \left[a_1, \frac{a+b}{2}\right].$$

Thus $[a_1, b_1]$ is a subinterval of $[a, b]$ with length $b_1 - a_1 = \frac{1}{2} l$, where $l([a, b]) = l$ such that $f(a_1) < 0 < f(b_1)$. Similarly one can construct a subinterval $[a_2, b_2]$ of $[a_1, b_1]$ with length $= \frac{b_1 - a_1}{2} = \frac{1}{2^2} l$ such that $f(a_2) < 0 < f(b_2)$. Inductively, we have a nested intervals $\{ [a_n, b_n] : n \in \mathbb{N} \}$ with ~~len~~ $b_n - a_n = \frac{l}{2^n}$ such that $f(a_n) < 0 < f(b_n)$. $\forall n \in \mathbb{N}$

Then $\exists\, \bar{x} \in [a,b]$ such that

$$\lim_n a_n = \bar{x} = \lim_n b_n$$

(why?). Since f is cts at $\bar{x}$ it follows (why?) that

$$\lim_n f(a_n) = f(\bar{x}) = \lim_n f(b_n)$$

(why?). By (??), $0 \leq \lim_n f(a_n)$ and $\lim_n f(b_n) \leq 0$

and so $f(\bar{x}) = 0$

Cor. Let $f : A \to \mathbb{R}$ be cts. $\emptyset \neq I \subseteq A$

$f(I) = \{f(x) : x \in I\}$

(i) If I is an interval then $f(I)$ is an interval

(ii) If I is a bounded closed interval then $f(I)$ is a bounded closed interval

(warning if $I = [a,b]$ it is <u>not</u> necessarily true that $f(I) = [f(a), f(b)]$.

(the correct ans should be:

$$f(I) = [f(x_*), f(x^*)]$$

where x_*, x^* are as in Th 1 (ii).

Proof (Hint for (i): $J \subseteq \mathbb{R}$ is an interval iff it is order-convex)